Mixed dynamics and tunneling

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(Received 12 June 1995)

A mechanism for the fluctuating features associated with the general phenomenon known as chaos-assisted tunneling is presented. These features are shown to be the result of a gradual competition between classically allowed (mixing) and classically forbidden (tunneling) dynamical processes. A random-matrix description of the phenomenon is provided. This requires, apart from the statistical ensembles commonly used to represent mixing, logarithmic-random ensembles to describe tunneling. Appearance and disappearance of tunneling partners and suppression of tunneling are predicted as possible features of chaos-assisted tunneling regions.

PACS number(s): 05.45.+b, 03.65.−w, 73.40.Gk

I. INTRODUCTION

The study of the effects produced by external perturbations on the dynamics of classically forbidden processes such as tunneling has become an active field of research. Most of the work has been done on bistable systems driven by time-dependent fields. The field conditions may be such that while negligible effects are observed on the system classical phase space, remarkable changes are produced in the tunneling dynamics; examples are the tunneling suppression phenomena under periodic forces [1] and magnetic fields [2].

A different class includes the cases in which the external perturbation changes considerably the system classical dynamics even to the point of producing widespread chaos. Some symmetry-related islands of Kol'mogorov-Arnol'd-Moser tori may still survive, becoming the loci of localized quantum states with tunneling splittings Δ orders of magnitude larger than those of the unperturbed system [3]. Several theoretical analyses relate this drastic change in the tunneling behavior to the important phase-space changes induced by the perturbation [3–6]. However, a clear and convincing mechanism that explains how the chaotic incoherent barrier crossing dynamics of the classically chaotic motion may affect so significantly the coherent tunneling dynamics of localized wave packets is still lacking. Recent work on this phenomenon shows a transition to chaos-assisted tunneling as the field strength varies, which takes place sharply when the localized tunneling-state wave functions start to overlap with the chaotic zones of the classical phase space [5]. The usual negative-slope linear dependence of $\log\Delta$ vs $1/\hbar$ disappears in this chaos-assisted tunneling region; instead, random looking oscillations are found with an average Δ practically independent of $\hbar$ [6,7].

In this paper we present simple matrix model Hamiltonians that provide a clear quantum mechanism for the fluctuating features observed in the chaos-assisted tunneling region. We demonstrate that these features are the result of a *gradual* competition between classically allowed (mixing) and classically forbidden (tunneling) dynamical processes.

II. A PHYSICAL SIMPLE MODEL

We start out by deducing, from physical arguments, a very simple model with the basic ingredients to display a fluctuating region in its tunneling behavior. Imagine a one-dimensional symmetric bistable system (e.g., a particle in a double-well potential), whose energy spectrum consists of tunneling pairs of different parity states. Together with tunneling, a second ingredient is needed: mixing. This is included by perturbing the system with an external oscillatory field in resonance with doublet-to-doublet transitions in a certain energy region. We can write then the following general Hamiltonian,

$$H = \sum_{i,\alpha} \varepsilon_{i_\alpha} b^\dagger_{i_\alpha} b_{i_\alpha} - E\cos(\omega t) \sum_{i,j,\alpha,\beta} M_{i_\alpha, j_\beta} b^\dagger_{i_\alpha} b_{j_\beta},$$

where we use the notation $|\, i_\alpha\rangle \equiv b^\dagger_{i_\alpha}$ and $\langle i_\alpha \,| \equiv b_{i_\alpha}$. Indices i and j label the doublet and take increasing integer values for increasing energies; α (or β) indicates the state symmetry (parity), either + or −. The amplitude of the driving field is denoted by E and its frequency by ω. M is the coupling matrix. If a symmetry is required in our perturbed system, M should be restricted to couple either equal parity (as in dipole interactions) or different parity (as in polarizability interactions) states.

Suppose now that E is small enough to perform a resonance approximation in H. This has two steps: truncation of the system to include only the states driven into resonance by the field, and time averaging over the remaining fast oscillations [8]. One finally arrives [8] at a low-field resonance Hamiltonian $H^{\rm res} = H_r + H_l + H_{rl}$, with

$$H_{r(l)} = \sum_i \varepsilon_i b^\dagger_{i_{r(l)}} b_{i_{r(l)}} - \frac{1}{2} E \sum_{i\neq j} V^{r(l)}_{ij} b^\dagger_{i_{r(l)}} b_{j_{r(l)}},$$

$$H_{rl} = -\frac{1}{2}\sum_i \Delta_i \left(b^\dagger_{i_r} b_{i_l} + b^\dagger_{i_l} b_{i_r} \right)$$
$$-\frac{1}{2} E \sum_{i\neq j} \left(W^r_{ij} b^\dagger_{i_r} b_{j_l} + W^l_{ij} b^\dagger_{i_l} b_{j_r} \right),$$

which is written in terms of "right" (r) and "left" (l) localized states $|i_r\rangle=\frac{1}{\sqrt{2}}(|i_+\rangle+|i_-\rangle)$, $|i_l\rangle=\frac{1}{\sqrt{2}}(|i_+\rangle-|i_-\rangle)$, with energies $\varepsilon_i=\frac{1}{2}(\varepsilon_{i_+}+\varepsilon_{i_-})-(i-1)\omega$. The parameters Δ_i, V_{ij}, and W_{ij} are renormalized splittings and couplings, which for small enough E can be approximated [8] by the bare values $\Delta_i=\varepsilon_{i_-}-\varepsilon_{i_+}$, $V^r_{ij}=\pm V^l_{ij}=(M_{i_+,j_\pm}+M_{i_-,j_\mp})(\delta_{i,j+1}+\delta_{i,j-1})$, $W^r_{ij}=\pm W^l_{ij}=(M_{i_+,j_\pm}-M_{i_-,j_\mp})(\delta_{i,j+1}+\delta_{i,j-1})$. The upper and lower signs in these expressions correspond, respectively, to equal-parity (EP) and different-parity (DP) M coupling matrices. Both Δ_i and W_{ij} couple r to l states and induce classically forbidden tunneling processes; V_{ij} couples r-r and l-l states inducing classically allowed mixing processes. In general $|V_{ij}|\gg|W_{ij}|\sim|\Delta_i\Delta_j|^{\frac{1}{2}}$.

Let us perform further simplifications. First, since the W_{ij} couplings are multiplied by the small field strength E in $H^{\rm res}$ we will neglect them against Δ_i. Second, the V_{ij} factors couple only next-neighbor states and their dependence on the state index i and on $\hbar$ in the resonance region is expected from semiclassical arguments to be weak; we will then choose $V^r_{ij}=\pm V^l_{ij}=V/E$. Finally, since only the i_r (i_l) states close in energy ε_i are mixed by V_{ij}, we will truncate the sums in $H^{\rm res}$ to include N r-l pairs and give them the same energy $\varepsilon_i=0$. All these simplifications lead, after changing the sign of the even index l states in the DP case, to the simple model

$$H^{\rm mod}=\frac{1}{2}V\sum_{i=1}^{N-1}\left(b^\dagger_{i_r}b_{(i+1)_r}\;+b^\dagger_{i_l}b_{(i+1)_l}+\text{H.c.}\right)$$
$$-\frac{1}{2}\sum_{i=1}^{N}(\pm1)^i\Delta_i\left(b^\dagger_{i_r}b_{i_l}+b^\dagger_{i_l}b_{i_r}\right), \qquad (1)$$

with two diagonal (r,l) blocks representing classically allowed mixing and two off-diagonal blocks representing quantum tunneling between r and l subsystems. H.c. stands for Hermitian conjugate. Eigenstates of $H^{\rm mod}$ are separated into (+) and (−) symmetry classes with respect to the transformation that interchanges (r) and (l) indices.

The different origin of the two coupling processes involved in $H^{\rm mod}$ suggests that the interplay between them is going to be determined by $\hbar$. Since Δ_i are the tunneling splittings for the unperturbed system, we will give them the semiclassical form $\Delta_i=A_i\,e^{-S_i/\hbar}$ whose more general validity has been established [9]. The prefactors A_i are smooth functions of $\hbar$ and S_i is the imaginary part of a classical action integral for a complex path [6,9]. Take now the energy region of the undriven system, which is mixed by the driving field to become the resonance zone, and call $S_{\rm min}$ and $S_{\rm max}$ the minimum and maximum values of S in such a region. If our system has only one degree of freedom and it is not too pathological, one expects S_i to increase monotonically from $S_{\rm min}$ to $S_{\rm max}$. If this increase is linear and the prefactor A_i is taken equal and independent of $\hbar$, then the tunneling splittings will be given by $\Delta_i=Ae^{-S_{\rm min}/\hbar}\,e^{-\alpha(i-1)}$, with $\alpha=(S_{\rm max}-S_{\rm min})/N\hbar$. The product $N\hbar$ is proportional to the phase-space volume of the resonance zone; thus α is independent of $\hbar$. Obviously, the number of states in the resonance zone N does depend on $\hbar$. Including this dependence in our model would complicate unnecessarily our analysis; thus we will fix N for the moment, which is equivalent to varying the field intensity in the form $Ee^{S_{\rm min}/\hbar}$, and scaling the results back to the initial value E. As will be demonstrated below the results obtained in this way are basically identical to those obtained with a true $\hbar$-dependent N.

III. RESULTS

Figure 1 presents, as a function of $S_{\rm min}/\hbar$, the tunneling splittings for our model with $N=10$, $\alpha=5$, $\ln(A)=25$ and $\ln(V)=-30$. Figures 1(a) and 1(b) correspond, respectively, to EP and DP coupling cases [upper and lower signs, respectively, in Eq. (1)]. A flat oscillatory transition region appears with many of the features observed in the chaos-assisted tunneling region of more complicated models [6,7]. The main difference is that, unlike the random nature of the fluctuations observed so far, the oscillations obtained here are quite regular. This is of course due to the simplicity of our model; we will see below how this model may be generalized to accommodate more realistic and complex situations. But let us concentrate for the moment on these results because this same simplicity will lead us to a transparent physical mechanism for the observed features.

Let us compare first the two limit situations $\hbar\to0$ and $\hbar\to\infty$. In the first case, the first term in Hamiltonian (1), which induces mixed dynamics, is just a negligible perturbation to the tunneling second term. Therefore, the tunneling behavior in this limit is that of the unper-

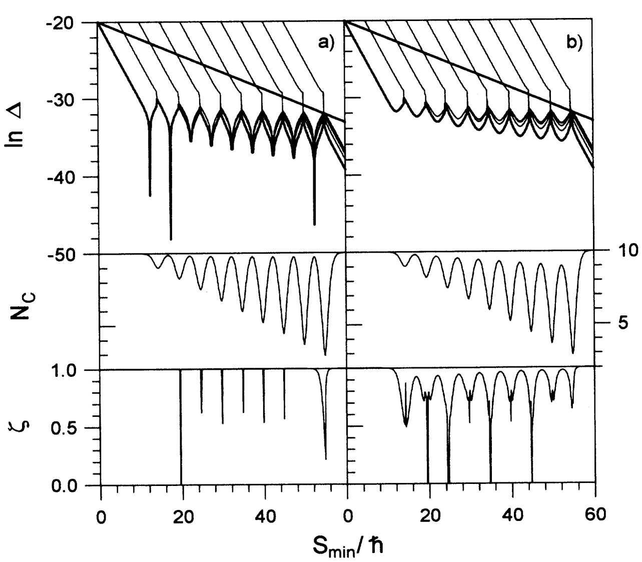

FIG. 1. Tunneling splittings (Δ), allowed maximum number of tunneling pairs (N_C), and pair detection reliability (ζ) for the Hamiltonian model in Eq. (1), as a function of $S_{\rm min}/\hbar$. (a) and (b) correspond, respectively, to equal parity and different parity couplings. In both cases ζ is calculated for the lowest splitting pair at each point. See text for details.

turbed system with the usual $1/\hbar$ negative-slope linear dependence. On the contrary, when $\hbar \to \infty$, mixing dominates and tunneling becomes a perturbation. The r and l states mix independently to produce linear superposition states in which every original state participates with similar probability. Perturbation theory gives then tunneling splittings that, due to our extreme mixing, get much closer to each other and closer to the highest possible value Δ_N. The tunneling dependence on $1/\hbar$ is again the usual linear one. These two limits appear in the figures.

Let us follow now the behavior of the model from the $\hbar \to 0$ to the $\hbar \to \infty$ limits. For large enough $\hbar$, Δ_N will get close in magnitude to the mixing coupling V, while the others $\Delta_{i\neq N}$ will still remain much smaller than V; this happens due to the Δ_i logarithmic-uniform distribution. At this point, the two states coupled by Δ_N will start to drop out the mixing process. As a consequence, the tunneling doublet corresponding to this pair acquires a splitting $\sim \Delta_N$, and the other still mixed pairs have their splittings reduced and bounded by the next highest $\Delta_i = \Delta_{N-1}$. This is just the first oscillation. The argument can be repeated for each pair of states; thus the number of oscillations will be $N-1$, the period α, and the average splitting in this region $\sim V$. The amplitude of the oscillation is also a function of α with some small differences among pairs; from the $N = 2$ case, which has an analytic solution, this amplitude was estimated to be $\sim \alpha - \alpha_c$; thus for α smaller than the critical value α_c the oscillation disappears, although a flat transition may still survive. We have obtained $\alpha_c \sim \ln 4$.

There is an important difference between the EP and DP coupling cases, namely, while no level crossing occurs between $(+)$ and $(-)$ states in the DP model, in the EP model all the $(+)$-$(-)$ pairs that remain mixed cross once every oscillation practically at the same $1/\hbar$ value. The crossings occur between tunneling partners and the minima in the oscillation of Fig. 1(a) correspond to these crossings. At the crossing points (r) and (l) localized states may be constructed for which tunneling is suppressed; even more, since the level crossings take place simultaneously for all mixed pairs, symmetry would be totally broken within this mixed subspace.

Above we mentioned that a correct study of the model as a function of $\hbar$ would involve an $\hbar$-dependent N. We can now demonstrate that, due to the gradual nature of the transition, an $\hbar$-dependent N is going to produce practically the same results as a fixed N calculation if N is properly chosen. Let us remember first that our tunneling splittings have the upper bound $\Delta_{\max} = Ae^{-S_{\min}/\hbar}$; thus, starting from the $\hbar \to 0$ limit, the transition region will begin when $\Delta_{\max} \sim V$. From these relations and our definition $\Delta_i = Ae^{-S_{\min}/\hbar}e^{-\alpha(i-1)}$, we can find the number of pairs $N = N_{\max}$ involved at this point in the transition, i.e., $N_{\max} \sim \frac{1}{\alpha}(1 - S_{\max}/S_{\min})\ln(A/V)$. Therefore, if one chooses to fix $N_{\max}$, then $\Delta_{\max}$ can be determined; in Figs. 1(a) and 1(b) the $\Delta_{\max}$ values for our 10-pair case are represented by thick straight lines. Now, a truly dependent $\hbar$ calculation would exclude at each point those pairs with tunneling splittings above these lines. However, the figures show clearly that these states have not been mixed yet and thus are not involved at all in the transition region; in other words, taking them out will not change anything except perhaps at the higher $1/\hbar$ end of the transition.

Mixing may involve states above the barrier, producing, in the classical limit, regions invariant against the r-l symmetry transformation. The Δ_i associated with these states would be practically independent of $\hbar$ and of the order of the mixing terms $V_{i,j}$. This situation corresponds to the particular case $S_{\min} = 0$, thus the number of states involved in the transition would be infinite, as expected, since barrier crossing persists in the classical limit.

There is a final aspect of the model of enough relevance to be discussed. It concerns the identification of the tunneling partner states, which until now we have assumed to be always possible. Tunneling partners should have different symmetry; let $|i_+\rangle$ and $|j_-\rangle$ be two normalized different-symmetry eigenstates of Hamiltonian (1); these two states are exact tunneling partners if $R_{ij} = |\langle i_+|\ \widehat{R}\ |j_-\rangle|^2 = 1$, where $\widehat{R}$ is the operator that changes the sign of all $|i_l\rangle$ states producing an opposite parity state. With this definition, tunneling partners do, of course, exist in our model in the two limits $\hbar \to 0$ and $\hbar \to \infty$. Elsewhere, only approximate tunneling pairs can be defined. Any measure of how close to either 0 or 1 R_{ij} matrix elements are could be used as a criterion to establish the adequacy of a description in terms of tunneling partners. Here, we have chosen to regard R as a channel matrix in the information theoretical sense [10]. The capacity C (in bits) of the channel is defined as its maximum mutual information [10] and $N_C = 2^C \leq N$ can be taken in our case as the maximum number of states in which one is allowed to look for their tunneling partners. The maximum mutual information is obtained for a given state-probability distribution p_i and $\zeta_i = p_i N_C$ is a measure of the maximum reliability $(\zeta \leq 1)$ of the process of detecting the tunneling partner for the state $|\,i_+\rangle$. All these quantities have been introduced as measures of stability, reliability, and complexity in quantum mechanics in Ref. [11]. Figures 1(a) and 1(b) include the values of N_C for our 10-pair system and those of ζ_i for the smallest splitting pair at each point. From these values we conclude that a description in terms of tunneling partners is inadequate around the oscillation maxima, while such a description is optimum at the oscillation minima. This would be consistent with and explain the observations made in some specific model Hamiltonians [6]. The fact that the crossings in our EP coupling case occur at regions where tunneling partners can be most easily identified gives definitive support to the possibility of localization within this model.

IV. A RANDOM-MATRIX MODEL

So far, we have been able to explain with our model all except one of the known features of chaos-assisted tunneling. This remaining feature is the random nature of the oscillation observed in some specific models with large phase-space regions of classical chaotic motion. Our model describes the most simple situation of an isolated resonance and the regularity of its mixing-tunneling transition is therefore not surprising. In more complicated situations energies and couplings will not be equal, different resonance zones may overlap, and other off-diagonal elements of both mixing and tunneling nature will be present. In such extreme mixing cases, where the classical motion is usually chaotic, the theory of random matrices is able to capture many of the features known as quantum chaos. Suppose we still use the structure of the Hamiltonian with the two symmetry related (r) and (l) subspaces.

$$H^{\text{mod}} = \frac{1}{2}\sum_{i,j=1}^{N} V_{ij}\left(b^{\dagger}_{i_r} b_{j_r} + b^{\dagger}_{i_l} b_{j_l}\right) - \frac{1}{2} e^{-S_{\text{min}}/\hbar} \sum_{i=1}^{N} A_i \Delta_i \left(b^{\dagger}_{i_r} b_{i_l} + b^{\dagger}_{i_l} b_{i_r}\right). \quad (2)$$

The diagonal form of Δ couplings in this expression is not a restriction since we can always arrive at such a form by a unitary transformation.

The matrix V_{ij} of $\hbar$-independent, classically allowed mixing processes is now taken from one of the random ensembles [12]. The exponential nature of the tunneling couplings points to a logarithmic-random form for Δ_i; this has been confirmed numerically in model Hamiltonians with chaotic classical limit and also by statistical analysis based on the structure of the classical phase space [6]. Thus we propose taking the values for Δ_i from a logarithmic-random ensemble, e.g., logarithmic exponential; the prefactors A_i may just be simply taken randomly as ± 1.

Figure 2 presents results of this random model for $N = 10$. The couplings V_{ij} were taken from the Gaussian orthogonal ensemble and Δ_i from a logarithmic-Gaussian ensemble. The features of the previous models remain

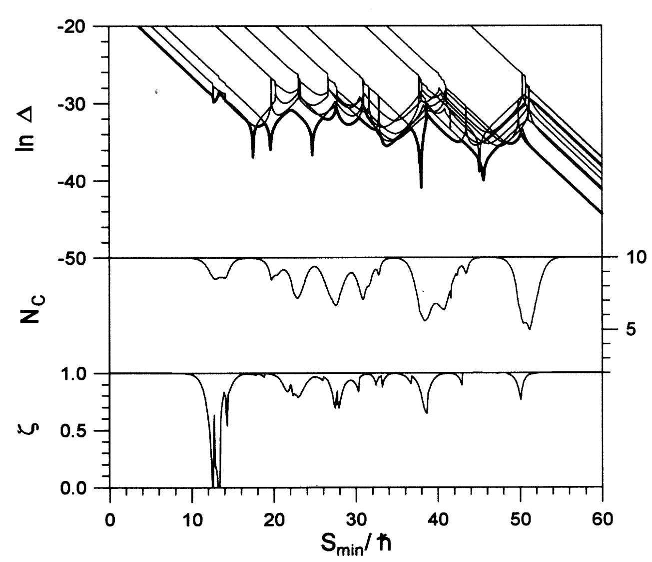

FIG. 2. Same as Fig. 1, for the random model Hamiltonian in Eq. (2). See text for details.

except for the now irregular oscillations. As before, at the oscillation maxima the tunneling doublet description fails, and exact localization is also possible at the level crossings of tunneling partners. Since there is not a constant α now, the size and amplitude of each oscillation change, although there is some correlation between them. The resemblance between Fig. 2 and the results obtained numerically for specific models [6,7] is rather good.

V. CONCLUSIONS

We have presented in this paper some simple matrix model Hamiltonians that display a chaos-assisted tunneling transition as a parameter $(\hbar)$ is varied. The first model has been deduced from physical arguments and corresponds to two symmetry related resonance zones coupled by exponentially small tunneling processes. Mixing and tunneling have been separated into two different terms in the Hamiltonian, where the role of $\hbar$ is to change their relative importance. We have shown then that the phenomenon known as chaos-assisted tunneling occurs at the transition from a region where classically allowed mixing processes smoothly dependent on $\hbar$ dominate, to a region in which classically forbidden tunneling processes take over. The logarithmic-uniform form of the tunneling couplings makes the transition occur gradually with regular oscillations every time the magnitude of a tunneling coupling gets close to the magnitude of the mixing coupling. Size and amplitude of these oscillations are correlated and the average splitting in the transition region has the order of magnitude of the mixing couplings. The maximum number of states involved in the transition and the number of oscillations are determined by the magnitude of the mixing couplings and the value of the maximum tunneling coupling allowed within the mixed phase-space region.

This matrix model was later generalized by introducing disorder in an attempt to accommodate more complex situations, such as those leading to the overlapping resonance mechanism responsible for chaotic motion in the classical limit. The proposed random-matrix model uses members of the Gaussian ensembles for the mixing term and members of logarithmic-random ensembles for the tunneling term. The random nature of these terms induces a random character in the fluctuating features of the chaos-assisted tunneling transition, with behaviors very reminiscent of those observed in specific systems. Finally, we have also seen that tunneling partners are not always defined and that tunneling suppression is a possible phenomenon in mixing-assisted tunneling.

In conclusion, the relative magnitude of mixing and tunneling terms, their distribution, and the number of states determines completely the features associated with the mixing-assisted tunneling transition displayed by our matrix models. How accurately can one map these models to specific systems? Recent calculations show that this mapping can be indeed established rather accurately at the qualitative level, providing good estimates of the transition region, tunneling splittings, number and size of the oscillations, and possibility of tunneling suppression [13].

ACKNOWLEDGMENTS

We thank Bruno Eckhardt for stimulating discussions. This work has been supported by grants from Ministerio de Educación y Ciencia (PB93-0578) and from Consejería de Educación del Gobierno de Canarias (92-077).

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